



TITLE:

Finite size approximation for  
representations of  
 $U_q(\widehat{\mathfrak{sl}}(n))$   
(Combinatorial Aspects in  
Representation Theory and  
Geometry)

AUTHOR(S):

Jimbo, Michio

---

CITATION:

Jimbo, Michio. Finite size approximation for representations of  $U_q(\widehat{\mathfrak{sl}}(n))$  (Combinatorial Aspects in Representation Theory and Geometry). 数理解析研究所講究録 1991, 765: 62-65

ISSUE DATE:

1991-08

URL:

<http://hdl.handle.net/2433/82283>

RIGHT:

# Finite size approximation for representations of $U_q(\widehat{\mathfrak{sl}}(n))$

神保道夫, 京大理

Michio Jimbo, Kyoto University

1. The present note is an elucidation of an observation made in [1] concerning the crystal base of integrable representations of  $U_q(\widehat{\mathfrak{sl}}(n))$ .

Let  $U_q = U_q(\widehat{\mathfrak{sl}}(2))$  denote the quantized affine algebra of type  $A_1^{(1)}$ . Just as in the classical case  $q = 1$ , it admits the following two classes of representations of particular interest:

- (1) Highest weight representations. These are irreducible modules  $L(\Lambda)$  with dominant integral highest weight  $\Lambda$ . For simplicity we consider here the level 1 representations  $L(\Lambda_i)$  ( $i = 0, 1$ ) where the  $\Lambda_i$  denote the fundamental weights.
- (2) Finite dimensional representations. These are level 0, non-highest weight representations (cf.[C]). For example, the natural representation  $V = \mathbb{C}^2$  of  $U_q(\mathfrak{sl}(2))$  can be made a  $U_q(\widehat{\mathfrak{sl}}(2))$ -module by letting the Chevalley generators act on  $V$  as follows:

$$e_0 = f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_1 = f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_0 = t_1^{-1} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix},$$

where  $t_i = q^{h_i}$ . (Here we follow the notations of [2]).

Given two modules  $L, L'$  over  $U_q$  one can form their tensor product  $L \otimes L'$  via the comultiplication

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i.$$

Let us consider the  $N$ -fold tensor product  $V^{\otimes N}$  of  $V = \mathbb{C}^2$ . Our objective here is to show the following fact

$$\lim_{N \rightarrow \infty} V^{\otimes N} \sim L(\Lambda_0) \sqcup L(\Lambda_1), \quad (*)$$

whose meaning will be made clear below.

2. The algebra  $U_q$  loses meaning at  $q = 0$ . However, Kashiwara's theory of crystal base [2] tells that on each integrable module  $L$  one can define the action of 'the Chevalley generators at  $q = 0$ '  $\tilde{e}_i, \tilde{f}_i$ . Moreover there exists a unique canonical base  $B = B(L)$  of  $L$  'at  $q = 0$ ', such that

$$\text{If } u, v \in B, \text{ then } \tilde{f}_i u = v \iff u = \tilde{e}_i v$$

holds. For precise statements see [2]. The above situation is represented as

$$u \xrightarrow{i} v.$$

This equips  $B$  with a structure of colored (by the index  $i = 0, 1$ ), oriented graph, called the crystal graph of  $L$ . It is known also that the crystal base  $B$  has a unique canonical extension to nonzero  $q$  [3].

There are some subtle points for finite-dimensional representations, since they are not integrable in the sense of [2]; but one can still consider crystal graphs for them. For instance  $V = \mathbb{C}^2$  has the crystal graph

$$u_0 \xrightleftharpoons[0]{1} u_1$$

with  $u_i$  denoting the natural base of  $V$ .

According to [2] the crystal graph behaves remarkably nicely under tensor products. The vertices of  $B(L_1 \otimes L_2)$  are simply  $B(L_1) \times B(L_2)$  as a set. The edges of the graph are described by a simple rule [2], color-by-color. It is an amusing exercise to work out the crystal graphs for  $B(V^{\otimes N})$  using this rule. Their vertices consist of sequences  $\xi = (\xi_1, \xi_2, \dots, \xi_N)$  with  $\xi_i \in \{0, 1\}$ , representing the vectors  $u_{\xi_1} \otimes \dots \otimes u_{\xi_N}$ . We show how they look like at the end of this note.

3. Let  $B_i^N$  ( $i = 0, 1$ ) be the full subgraph of the crystal graph  $B(V^{\otimes N})$ , whose vertices consist of sequences  $\xi = (\xi_1, \xi_2, \dots, \xi_N)$  with  $\xi_N = i$ . From the figure for  $N = 2, 3, 4$  the following is already apparent:

**Theorem.** *There is an imbedding of graphs  $B_i^N \hookrightarrow B_{i+1}^{N+1}$  given by  $v \mapsto v \otimes u_{i+1}$ , where the suffix  $i$  is to be read modulo 2. As  $N$  even  $\rightarrow \infty$ ,  $B_i^N$  converges to the crystal graph  $B(L(\Lambda_i))$  of the highest weight representation  $L(\Lambda_i)$  (with the arrows reversed, because of conventions).*

Thus the equality (\*) makes sense in the language of crystal base. The proof of the theorem can be done by straightforward induction using Kashiwara's rule. As a consequence,  $L(\Lambda_i)$  has a basis labeled by infinite sequences (called paths)

$\xi = (\xi_1, \xi_2, \dots)$ , whose 'tail' is  $\dots 010101 \dots$  (i.e.  $\xi_j \equiv j + i - 1 \pmod{2}$  for  $j \gg 0$ ). Though we have omitted here, there is also a formula for the weight of these base vectors given in terms of the paths [1]. This type of result has an important application in solvable lattice models of statistical mechanics [4]; in fact the whole story was motivated by the latter.

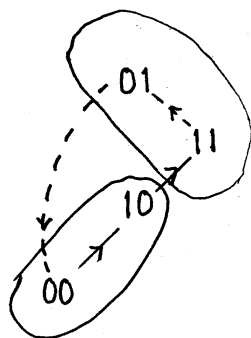
4. In [1] a similar result is established for  $U_q(\widehat{\mathfrak{sl}}(n))$ . Integrable representations of arbitrary level  $l$  can be 'approximated' by taking  $V = S^l(\mathbb{C}^n)$ , the  $l$ -th symmetric power of the standard representation  $\mathbb{C}^n$ .

*Remark.* At the stage of writing this note, Kashiwara found a simple explanation to this phenomenon.

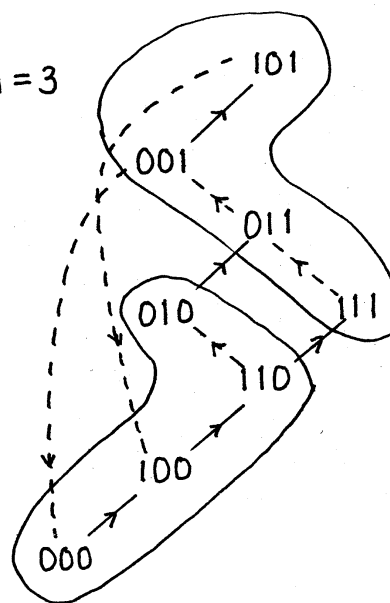
### References

- [1] M. Jimbo, K. C. Misra, M. Okado and T. Miwa, Combinatorics of representations of  $U_q(\widehat{\mathfrak{sl}}(n))$  at  $q = 0$ , preprint RIMS 709 (1990)
- [C] V. Chari, Integrable representations of affine Lie algebras, *Invent. Math.* **85** (1986) 317–335.
- [2] M. Kashiwara, Crystalizing the  $q$ -analogue of universal enveloping algebras, to appear in *Commun. Math. Phys.*
- [3] M. Kashiwara, On crystal bases of the  $q$ -analogue of universal enveloping algebras, RIMS preprint, Kyoto Univ. 1990.
- [4] See e.g. the review 'Solvable Lattice Models', *Proceedings of Symposia in Pure Mathematics*, **49** (1989) 295–331.

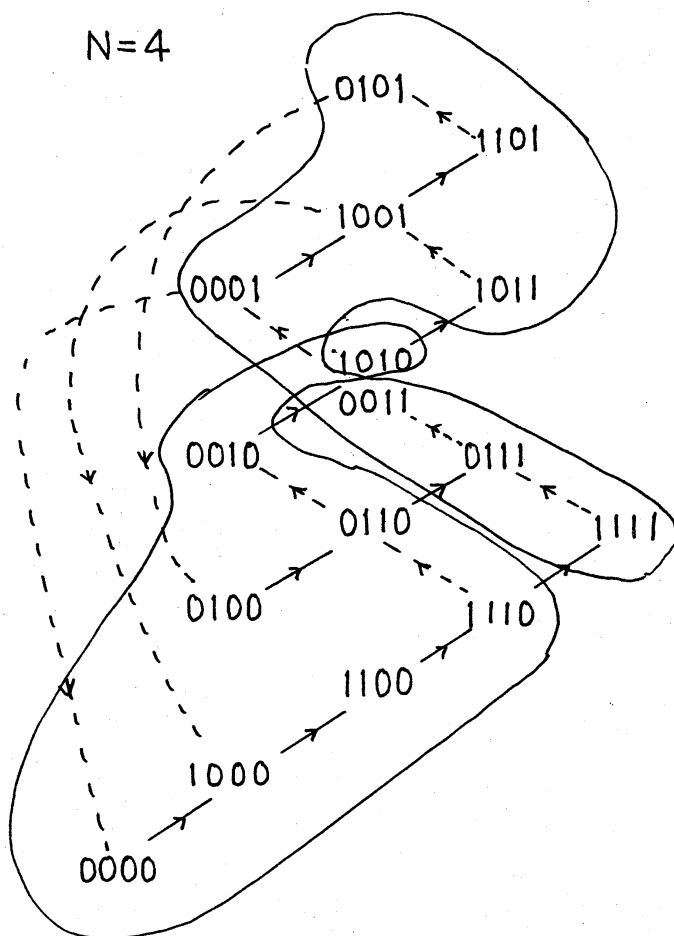
N=2



N=3



N=4



< Convergence of crystal  
graphs for  $V^{\otimes N}$  >

$$\longrightarrow = \frac{1}{\longrightarrow}$$

$$\dashrightarrow = \frac{0}{\longrightarrow}$$

$$0110 = u_0 \otimes u_1 \otimes u_1 \otimes u_0 \text{ etc.}$$